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On almost complex curves in the nearly Kaehler six-sphere ☆

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ABSTRACT

In this paper, we study the curvature properties of almost complex curves M in the nearly Kaehler six-sphere by using the harmonic sequences theory. For compact almost complex curve of type (I), if the Gaussian curvature $K \leqslant \frac{1}{6}$, then $K = \frac{1}{6}$. A basic valued distribution theorem of Gaussian curvature for almost complex curve of type (II) is given. For almost complex curve of type (III), we show that if M is complete and Gaussian curvature $K \geqslant 0$, then $K = 0$; and if M is compact and $K \leqslant 0$, then $K = 0$.

1. Introduction

An almost complex curve in S^6 is a non-constant smooth immersion $f : M^2 \to S^6$, from a Riemann surface M^2, whose differential is complex linear. It is an elementary fact that such a map is a conformal harmonic map. A major motivation for studying almost complex curve in S^6 is that the cone on such a curve is absolutely volume minimizing in $\mathrm{Im}\,\mathbf{O} \cong R^7$ [8]. Bolton, Vrancken and Woodward [3] showed that the almost complex curves in S^6 are divided into four types: (I) linearly full in S^6 and superminimal; (II) linearly full in S^6 and not superminimal; (III) linearly full in some totally geodesic S^5 in S^6; (IV) totally geodesic.

For type (I), Bryant [5] gave the construction by using the twistor method, and proved that there are compact almost complex curves of type (I) of every genus. For type (II) and type (III), Bolton, Pedit and Woodward [4] showed that such an almost complex immersion f can be lifted to $\tilde{f} : M^2 \to G_2/T^2$, where T^2 is the maximal torus of $SU(3)$. With respect to the curvature problem of the almost complex curve, Sekigawa [12] obtained that if the Gaussian curvature K of the almost complex curve is constant, then $K = 0, \frac{1}{6}, 1$. Dillen, Verstraelen and Vrancken [7] showed that if M is a compact almost complex curve in S^6 with Gaussian curvature K, (i) if $\frac{1}{6} \leqslant K \leqslant 1$, then either $K \equiv \frac{1}{6}$ or $K \equiv 1$; (ii) if $0 \leqslant K \leqslant \frac{1}{6}$, then either $K \equiv 0$ or $K \equiv \frac{1}{6}$.

The paper is organized as follows. In Section 2, we give some fundamental formulas and facts of the harmonic sequences associated to almost complex curves in S^6. In Section 3, we prove our main results, Theorem 3.1, Theorem 3.3 and Theorem 3.4.

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2. Preliminaries

It is well known that the 6-dimensional sphere S^6 admits an almost complex structure J by using the algebra of Cayley numbers, and together with the standard metric on S^6, this almost complex structure J is a nearly Kaehler structure. S^6 is a typical example of nearly Kaehler manifold. For details, refer to [8]. In the following, our notations with respect to harmonic sequences theory are the same as in [1–3].

Let $f : M \to S^6$ be an almost complex curve, then in terms of a local complex coordinate $z = x + \sqrt{-1}y$, the harmonic sequence of f is denoted by $\{[f_p]\}_{p\in Z}$. We have

$$\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right) = \left(\frac{\partial}{\partial y}, \frac{\partial}{\partial y}\right), \qquad \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) = 0, \tag{2.1}$$

$$II\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) = J \cdot II\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right), \tag{2.2}$$

where $(\cdot,\cdot)$ is the induced metric on M, in fact, the induced metric is $g_0 = 2\gamma_0|dz|^2$, J is the standard nearly Kaehler structure on S^6, II is the second fundamental form of f.

From (2.1), (2.2), it follows that f is a conformal harmonic map. We see that f is an almost complex curve if and only if

$$f_0 \times f_1 = \sqrt{-1} f_1. \tag{2.3}$$

If we differentiate (2.3), and again, with respect to z, we obtain

$$f_0 \times f_2 = \sqrt{-1} f_2, \qquad f_1 \times f_2 + f_0 \times f_3 = \sqrt{-1} f_3. \tag{2.4}$$

With respect to the second fundamental form II of f, and its complex linear extension, we have

$$II\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial z}\right) = f_2 = \frac{1}{2}\left[II\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right) - \sqrt{-1} II\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)\right]. \tag{2.5}$$

The third fundamental form III of f is defined by [3]

$$III(X, Y, Z) = \big(\mathbf{D}_X II(Y, Z)\big)^{\perp},$$

where $\mathbf{D}$ denotes the flat connection on R^7, and $(\cdot)^{\perp}$ is the component perpendicular to f_0, $\operatorname{Re} f_1$, $\operatorname{Im} f_1$, $\operatorname{Re} f_2$, $\operatorname{Im} f_2$. Then it follows

$$III\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial z}, \frac{\partial}{\partial z}\right) = f_3 = \frac{1}{2}\left[III\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right) - \sqrt{-1} III\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)\right]. \tag{2.6}$$

From (2.2) and (2.5), we obtain the square of length of the second fundamental form II of f

$$|II|^2 = 2\frac{|f_2|^2}{|f_1|^4} = 2\frac{\gamma_1}{\gamma_0}. \tag{2.7}$$

By (2.10) in [1], it follows

$$\Delta \log\big(|II|^2\big) = \frac{2}{\gamma_0}(\gamma_2 - 3\gamma_1 + 2\gamma_0), \tag{2.8}$$

$$\Delta |II|^2 = 4\frac{\gamma_1}{\gamma_0^2}(\gamma_2 - 3\gamma_1 + 2\gamma_0) + 4\big|\nabla |II|\big|^2. \tag{2.9}$$

The Gaussian equation of the almost complex curve f is

$$K + \frac{|II|^2}{2} = 1. \tag{2.10}$$

Remark 2.1. Setting aside the trivial case (IV), almost complex curve in the nearly Kaehler S^6 is either superminimal or superconformal. If the second fundamental form II of f is parallel, then $II = 0$, i.e., f is totally geodesic.

Remark 2.2. We should remark that Jiao and Peng [11] showed the similar useful formula as (2.8) for a linearly full totally real conformal minimal immersion of S^2 into CP^n.

3. Proof of the main result

In this section, we study the curvature properties of different types of almost complex curve in S^6. Meanwhile, we give the integrability conditions for different almost complex curves.

3.1. Almost complex curve of type (I)

Let $f : M^2 \to S^6$ be a linearly full superminimal almost complex curve, $z = x + \sqrt{-1}y$ is a local complex coordinate on Riemann surface M. The harmonic sequence of f is

$$0 \to [f_{-3}] \to [f_{-2}] \to [f_{-1}] \to [f_0 = f] \to [f_1] \to [f_2] \to [f_3] \to 0,$$

where $[f_3] : M \to CP^6$ is a linearly full and anti-holomorphic map, and $[f_{-3}] : M \to CP^6$ is a linearly full and holomorphic map. We have

$$f_0 \times f_3 = -\sqrt{-1} f_3, \qquad f_1 \times f_2 = 2\sqrt{-1} f_3. \tag{3.1}$$

Since

$$|f_1 \times f_2|^2 = 2|f_1|^2|f_2|^2,$$

it follows

$$|f_1|^2|f_2|^2 = 2|f_3|^3, \qquad \gamma_0 = 2\gamma_2. \tag{3.2}$$

In fact, we can choose G_2-frame along f as in [3], and the integrability condition is

$$2\frac{\partial^2 w_1}{\partial z \partial \bar{z}} = e^{2(w_2 - w_1)} - e^{2w_1}, \qquad 2\frac{\partial^2 w_2}{\partial z \partial \bar{z}} = e^{2(w_3 - w_2)} - e^{2(w_2 - w_1)}, \qquad 2\frac{\partial^2 w_3}{\partial z \partial \bar{z}} = -e^{2(w_3 - w_2)},$$

where $|f_i|^2 = e^{2w_i}$, $i = 1, 2, 3$.

Theorem 3.1. *Let $f : M^2 \to S^6$ be a linearly full superminimal compact almost complex curve with Gaussian curvature K.*

(i) *If $|II|^2 \leqslant \frac{5}{3}$ (i.e., $K \geqslant \frac{1}{6}$) everywhere on M, then $K = \frac{1}{6}$;*
(ii) *If $|II|^2 \geqslant \frac{5}{3}$ (i.e., $K \leqslant \frac{1}{6}$) everywhere on M, then $K = \frac{1}{6}$.*

Proof. From (2.8), (3.2), we obtain

$$\triangle \log\bigl(|II|^2\bigr) = 5 - 6\frac{\gamma_1}{\gamma_0}.$$

Since $\frac{\gamma_1}{\gamma_0} = \frac{|II|^2}{2}$, then $\triangle \log(|II|^2) = 5 - 3|II|^2$.

If $|II|^2 \leqslant \frac{5}{3}$, then $\triangle \log(|II|^2) \geqslant 0$. By the well-known E. Hopf's maximum principle, we obtain $|II|^2 = \text{constant}$, thus $|II|^2 = \frac{5}{3}$. By the Gaussian equation (2.10), $K = \frac{1}{6}$.

(ii) follows similar from (2.8), (3.2) and the maximum principle of E. Hopf. □

Remark 3.2. In the above theorem, if Gaussian curvature $K \geqslant \frac{1}{6}$, then by Gauss–Bonnet Theorem, $M \cong S^2$, the result (i) of Theorem 3.1 was also proved in [1,7] by using some different methods.

3.2. Almost complex curve of type (II)

Let $f : M^2 \to S^6$ be a linearly full superconformal almost complex curve, $z = x + \sqrt{-1}y$ is a local complex coordinate on Riemann surface M. The harmonic sequence of f is

$$\cdots \to [f_{-3}] \to [f_{-2}] \to [f_{-1}] \to [f_0 = f] \to [f_1] \to [f_2] \to [f_3] \to \cdots,$$

where every 6 consecutive elements of the sequence $\{[f_p]\}_{p \in \mathbf{Z}}$ are mutually orthogonal, but not periodic, f_{-3} and f_3 are linearly independent.

Set

$$\frac{|(f_3, f_3)|}{|f_3|^2} = \frac{1}{\cosh 2\eta}, \tag{3.3}$$

then obviously, the real valued function η is a globally defined function on M [3], and $\eta > 0$ or $\eta < 0$. Here, $(\cdot,\cdot)$ denotes the extension of the standard inner product in R^7 to a symmetric complex bilinear form on C^7.

Since (f_3, f_3) is a holomorphic function, and $(f_3, f_3) \neq 0$, we may choose a local complex coordinate z (also denoted by z), such that $(f_3, f_3) = 1$. Hereafter, we assume $\eta > 0$. Choosing G_2-frame along f as in [3], and the integrability condition is

$$2\frac{\partial^2 w_1}{\partial z\partial\bar{z}} = e^{2(w_2-w_1)} - e^{2w_1}, \qquad \frac{\partial^2\eta}{\partial z\partial\bar{z}} = -\sinh\eta\cosh\eta e^{-2w_2},$$
$$2\frac{\partial^2 w_2}{\partial z\partial\bar{z}} = \left(\sinh^2\eta + \cosh^2\eta\right)e^{-2w_2} - e^{2(w_2-w_1)}, \qquad w_1 + w_2 = \eta,$$

where $|f_i|^2 = e^{2w_i}$, $i = 1, 2$.

For M is compact, we define

$$M_\pm = \max_M \frac{1}{1\pm\tanh 2\eta}, \qquad m_\pm = \min_M \frac{1}{1\pm\tanh 2\eta}. \tag{3.4}$$

Now, we state the following valued distribution theorem of Gaussian curvature.

Theorem 3.3. *Let $f : M^2 \to S^6$ be a linearly full superconformal compact almost complex curve with Gaussian curvature K. Suppose $|II| \neq 0$ everywhere on M.*

(i) *If $t_1t_2 > 1$ everywhere on M, then there must exist some points $p_1, p_2 \in M$, satisfying $K(p_1) > \frac{1-M_+}{3}$, $K(p_2) < \frac{1-m_+}{3}$;*
(ii) *If $t_1t_2 < 1$ everywhere on M, then there must exist some points $q_1, q_2 \in M$, satisfying $K(q_1) > \frac{1-M_-}{3}$, $K(q_2) < \frac{1-m_-}{3}$,*

where t_1 and t_2 are the Kaehler angle functions defined in [1].

Proof. By Lemma 5.4 in [3] and (3.3), we have

$$\left(\frac{\gamma_0}{\gamma_2} - 1\right)^2 = \tanh^2 2\eta. \tag{3.5}$$

Thus

$$\frac{\gamma_0}{\gamma_2} = 1 \pm \tanh 2\eta, \quad \text{i.e.,}\ t_1t_2 = 1 \pm \tanh 2\eta.$$

From (2.7), (2.8), (2.10), we get

$$\triangle \log\left(|II|^2\right) = 6K + 2\frac{\gamma_2}{\gamma_0} - 2, \tag{3.6}$$

i.e.,

$$\triangle \log\left(|II|^2\right) = 6K + \frac{2}{1\pm\tanh 2\eta} - 2. \tag{3.7}$$

Then, from (3.4), it follows

$$\triangle \log\left(|II|^2\right) \leqslant 6K + 2M_\pm - 2, \tag{3.8}$$
$$\triangle \log\left(|II|^2\right) \geqslant 6K + 2m_\pm - 2. \tag{3.9}$$

(i) $t_1t_2 > 1$, corresponding to M_+ and m_+, then if the Gaussian curvature $K \leqslant \frac{1-M_+}{3}$ everywhere on M, or $K \geqslant \frac{1-m_+}{3}$ everywhere on M, by (3.8), (3.9), and E. Hopf's maximum principle, we obtain $|II|^2 =$ constant, then $K =$ constant, but this is impossible, because there does not exist almost complex curves of type (II) with constant curvature [6]. It follows that there must exist some points $p_1, p_2 \in M$, satisfying $K(p_1) > \frac{1-M_+}{3}$, $K(p_2) < \frac{1-m_+}{3}$.

(ii) $t_1t_2 < 1$, corresponding to M_- and m_-, then if the Gaussian curvature $K \leqslant \frac{1-M_-}{3}$ everywhere on M, or $K \geqslant \frac{1-m_-}{3}$ everywhere on M, by (3.8), (3.9), and E. Hopf's maximum principle, we obtain $|II|^2 =$ constant, then $K =$ constant, but this is impossible, for there does not exist almost complex curves of type (II) with constant curvature [6]. It follows that there must exist some points $q_1, q_2 \in M$, satisfying $K(q_1) > \frac{1-M_-}{3}$, $K(q_2) < \frac{1-m_-}{3}$. $\square$

3.3. Almost complex curve of type (III)

Let $f : M^2 \to S^5_N \hookrightarrow S^6$ be an almost complex curve in some totally geodesic S^5_N, N is a constant vector in R^7 orthogonal to S^5_N, let $z = x + \sqrt{-1}y$ be a local complex coordinate on Riemann surface M. The harmonic sequence of f is

$$\cdots \to [f_{-3}] \to [f_{-2}] \to [f_{-1}] \to [f_0 = f] \to [f_1] \to [f_2] \to [f_3] \to \cdots,$$

where every 6 consecutive elements of the sequence $\{[f_p]\}_{p\in\mathbf{Z}}$ are mutually orthogonal, and periodic, i.e., $\psi_{-3} = [f_{-3}] = [f_3] = \psi_3 : M \to CP^6$.

As in type (II), since (f_3, f_3) is a holomorphic function, and $(f_3, f_3) \neq 0$, we can choose a local complex coordinate z (also denoted by z), such that $(f_3, f_3) = 1$. In fact, $f_3 = \frac{1}{2} III(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}, \frac{\partial}{\partial x})$. Choosing G_2-frame along f as in [3], and the integrability condition is

$$2\frac{\partial^2 w_1}{\partial z \partial \bar{z}} = e^{2(w_2 - w_1)} - e^{2w_1}, \qquad -\frac{\partial w_1}{\partial z} = \frac{\partial w_2}{\partial z},$$

where $|f_i|^2 = e^{2w_i}$, $i = 1, 2$.

Theorem 3.4. *Let $f : M^2 \to S^6$ be an almost complex curve of type* (III)*, its Gaussian curvature denoted by K.*

(i) *If M is complete and $K \geqslant 0$, then $K = 0$;*
(ii) *If M is compact and $K \leqslant 0$, then $K = 0$.*

Proof. For almost complex curve of type (III), we have

$$|f_1 \times f_2|^2 = 2|f_1|^2|f_2|^2, \qquad |f_1 \times f_2|^2 = 2|f_3|^2,$$

it follows

$$|f_1|^2|f_2|^2 = |f_3|^2, \qquad \gamma_0 = \gamma_2. \tag{3.10}$$

Then together with (2.8) and (2.9), we obtain

$$\Delta \log\bigl(|II|^2\bigr) = 6 - 6\frac{\gamma_1}{\gamma_0} = 6K, \tag{3.11}$$

$$\Delta|II|^2 = 6K|II|^2 + 4\bigl|\nabla|II|\bigr|^2. \tag{3.12}$$

If $K \geqslant 0$, from (2.10) and (3.12), it follows that $|II|^2$ is a bounded subharmonic function on M^2. A theorem of Blanc, Fiala and Huber [10] states that every complete non-negatively curved Riemann surface is parabolic. Hence $|II|^2 =$ constant, and then $K = 0$.

(ii) follows from (3.11) and the well-known E. Hopf's maximum principle. $\square$

Remark 3.5. Recently, we found that the result (i) of Theorem 3.4 was also proved by Udagawa [13] by using a different method.

Remark 3.6. By the Kaehler angle formula (4.3) in [1] of the harmonic sequence $\{[f_p]\}$, and together with (3.2), (3.5), (3.10), we get for almost complex curve of type (I), $t_1 \cdot t_2 = 2$; for almost complex curve of type (II), $0 < t_1 \cdot t_2 < 2$; for almost complex curve of type (III), $t_1 \cdot t_2 = 1$.

Remark 3.7. As showed in [4], almost complex curves of type (II) and (III) correspond to solutions of the affine Toda field model for G_2. In theory, many nontrivial examples of type (II) and (III) exist. Finding the nontrivial examples is an interesting problem. For example, Hashimoto, Taniguchi and Udagawa [9] showed a nontrivial example of almost complex two-tori of type (III) in terms of the Jacobian elliptic functions.

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